eigenvalues of $\bar{A}$ only have one pole each and therefore no roots appear for finite ω. If we allow more than one pole in the eigenvalues of $\bar{A}$, we are able to attain, by correct choice of the poles and residues, that the number of poles of $P\,\bar{\Gamma}^{\omega}\,P$ diminish. However, the danger lies in the fact that by an incorrect choice of the poles and residues the effective interaction $P\,\bar{\Gamma}^{\omega}\,P$ obtains additional poles.

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Representation of Symmetries and Observables in Functional Hilbert Space. II. *

A. Rieckers

Institut für Theoretische Physik der Universität Tübingen

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The representation of infinitesimal generators corresponding to the group representation discussed in the preceding paper is analyzed in the Hilbert space of functionals over real test functions. Explicit expressions for these unbounded operators are constructed by means of the functional derivative and by canonical operator pairs on dense domains. The behaviour under certain basis transformations is investigated, also for non-Hermitian generators. For the Hermitian ones a common, dense domain is set up where they are essentially selfadjoint. After having established a one-to-one correspondence between the real test function space and a complex Hilbert space the theory of quantum observables is applied to the functional version of a relativistic quantum field theory.

1. Introduction

In the preceding article [1] (Part I) two isomorphic realizations $\mathscr{L}^2(\mathscr{H}_r, \gamma)$ and $\mathscr{L}^2(\mathscr{H}_r, \lambda)$ of the Hilbert space of all square-integrable functionals on the real test function space $\mathscr{H}_r$ have been introduced. Their scalar products are constructed by means of the Friedrich-Shapiro integral where the integration space $\mathscr{H}_r$ is itself assumed to be a Hilbert space, and the characters γ and λ remind us of the Gaussian and Lebesgues measures. In the functional Hilbert space a group representation was investigated which comes into play in the functional formulation of quantum field theory (cf., also [2]). This kind of group representation is generated by an operator-function $\pi(U)$ (resp. by its associated variants $\bar{\pi}(U)$, $\pi_\lambda(U)$) which maps operators U of the test function space homomorphic with respect to the operator-multiplication into operators of the functional space.

In this paper we are concerned with the infinitesimal generators of the mentioned group representation. They are induced into the functional Hilbert space via a linear operator-function $\sigma(A)$ [resp. $\bar{\sigma}(A)$, $\sigma_\lambda(A)$] which maps an operator A of the test function space homomorphic with respect to the commutator-composition into a functional operator. It is shown that $\sigma(A)$ is in any case an unbounded operator whatever well-behaved A may be. Thus, all statements on $\sigma(A)$ have to be specified on which domains they are valid. Also for unbounded operators A it is possible to express $\sigma(A)$ explicitly in virtue of so called cononical operator pairs, which are analogous to the creation and annihilation operators of Fock space. In spite of the structural analogy of the functional Hilbert space and the Fock space which was also emphasized by Segal [3] one should not be mistaken to consider the underlying field theory as a trivial one. There are also structural deviations from Fock space

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Reprint requests to Dr. A. Rieckers, Institut für Theoretische Physik der Universität Tübingen, *D-7400 Tübingen*, Wächterstraße 67.

which are indicated by the emergence of at least three natural group-representing operator functions in comparison with the single one in Fock space. Having expressed $\sigma(A)$ by strongly converging, infinite sums over the canonical operator pairs one is able to study the **R**- and **V**-transformed functions $\bar{\sigma}(A)$ and $\sigma_\lambda(A)$. In contrast to $\bar{\pi}(U)$ and $\pi_\lambda(U)$ one obtains rather concise expressions for the infinitesimal generators. In course of this investigation also the transformed canonical operator pairs are evaluated showing beside other things the connection of our second realization of the functional Hilbert space with the algebraic construction of STUMPF[4]. For $\bar{\sigma}(A)$ and $\sigma_\lambda(A)$ the general *-homomorphism property with respect to Hermitian conjugation is proved. For anti-Hermitian A all three operator-functions coincide. A rather large domain is given, where $i\sigma(A)$ is then an essentially-selfadjoint operator. This domain, on which also other parts of the analysis are carried through, contains states of an infinite number of particles.

The usual situation in quantum mechanics that there is given a representation of the symmetries and observables in a complex test function space, the test functions being nothing but the ordinary wave functions, is taken as starting point in the last chapter. From there the transition into an equivalent real Hilbert space is performed in quite general terms. This enables us to transcribe the functional formulation of a non-Hermitian field theory into a Hermitian one. Discussions in which way the functional observables characterize the states in a relativistic quantum field theory finish our investigations.

Let us collect here some notations for later use:

$\mathscr{H}_r$ is the real Hilbert space of test functions, in which the scalarproduct (j, h) and the norm $\|j\|_r$ is defined for $j, h \in \mathscr{H}_r$. $\mathscr{H}_c$ denotes the complex extension $\mathscr{H}_r + i\mathscr{H}_r$.

The functional Hilbert space $\mathscr{L}^2(\mathscr{H}_r, \gamma)$ has the inner product $\langle \mathfrak{T}, \mathfrak{S} \rangle$ and the norm

$$\|\mathfrak{T}\|, \mathfrak{T}, \mathfrak{S} \in \mathscr{L}^2(\mathscr{H}_r, \gamma).$$

The domain of definition of an operator A is denoted by $\mathscr{D}[A]$.

2. Representation of the Infinitesimal Generators of $\pi(U)$

For the investigation of infinitesimal generators in the functional Hilbert space we draw our attention to that operator-function which maps an infinitesimal semigroup generator of the test function space into an infinitesimal semigroup generator of the functional Hilbert space. Let A be a densely defined operator in $\mathscr{H}_r$. A is called a semigroup generator if there exists a weakly continuous, one-parametric operator-function U_t, $t \in [0, \infty)$ with $U_0 = 1$, $U_t U_{t'} = U_{t+t'}$ so that

$$A = \lim_{t \to 0} \frac{1}{t} [U_t - 1].$$

It is surprising that we need in the last limit only to assume weak convergence. The condition that A generates a semi-group is thus a rather weak assumption. From Proposition 3.2 of Part I we know that $\pi(U_t)$ is a strongly continuous semigroup of densely defined operators in the functional Hilbert space $\mathscr{L}^2(\mathscr{H}_r, \gamma)$. According to [5], p. 307, it exists therefore a unique, densely defined operator in $\mathscr{L}^2(\mathscr{H}_r, \gamma)$ which generates $\pi(U_t)$. Thus, the following definition makes sense.

Definition 2.1. Let A be the densely defined generator of a semigroup U_t in $\mathscr{H}_r$, then we denote by $\sigma(A)$ that operator-function which maps A into the generator of $\pi(U_t)$.

One could try to get some information on $\sigma(A)$ by abstract semigroup theory. But we prefer to derive explicit expressions for this operator-function and to discuss then its properties. In this way we obtain much stronger results. Explicit representations of $\sigma(A)$ are gained by the aid of so-called canonical operator-pairs.

Definition 2.2. The pair of operator-valued functionals $C(g)$, $C^*(f)$, $g, f \in \mathscr{H}_c$, is said to be canonical if the commutation relations

$$[C(g), C(f)] = [C^*(g), C^*(f)] = 0, \qquad (2.1)$$
$$[C(g), C^*(f)] = (f, g)$$

are satisfied on a dense domain in $\mathscr{L}^2(\mathscr{H}_r, \gamma)$.

The first canonical operator-pair occuring in our investigation consists of the functional derivative and of the operation of multiplication by a linear functional.

Definition 2.3. For each $h \in \mathscr{H}_r$ the pointwise limit

$$[(h, \delta/\delta j)\, \mathfrak{T}](j) := (h, \delta/\delta j)\, \mathfrak{T}(j) := \lim_{\varepsilon \to 0} [\mathfrak{T}(j + \varepsilon h) - \mathfrak{T}(j)]$$

is called, if it exists, the functional derivative (Gateaux-derivative) of $\mathfrak{T}(j)$ in the direction of h (see [5], p. 110).

In general $(h, \delta/\delta j)$ does not possess very pleasing features since it need not commute with itself for different vectors h and is not necessarily linear in h (see [6], p. 38). Therefore, we apply the functional derivative only on continuously differentiable functionals $\mathfrak{T}(j)$, i.e., for which the inequality

$$|(h, \delta/\delta j)\, \mathfrak{T}(j)| \leqq \mathfrak{S}(j)\, \|h\|_{\mathrm{r}} \tag{2.3}$$

is valid, where the positive functional $\mathfrak{S}(j)$ does not depend on h. Stated otherwise, we generally restrict our considerations to those functionals where $(h, \delta/\delta j)$ equals the Volterra-derivative (see [5], p. 110). In fact, the Volterra-derivative does commute with itself and is linear in the increment, i.e.,

$$(\alpha_1 h_1 + \alpha_2 h_2, \delta/\delta j) = \alpha_1 (h_1, \delta/\delta j) + \alpha_2 (h_2, \delta/\delta j), \tag{2.4}$$

where $\alpha_1, \alpha_2 \in R$. There is a dense set of functionals including the set of all polynomials on which Def. 2.3 is meaningful for complex vectors h, too. These functionals are called holomorphic. It is easy to show, that

$$[(g, \delta/\delta j), (f, j)] = (f, g) \tag{2.5}$$

proving that $(g, \delta/\delta j)$ and (f, j) constitute a canonical operator-pair.

Let us derive some properties of differentiable, tame functionals. If $\mathfrak{T}(j)$ is differentiable and is based on the subspace $\mathscr{H}_{\mathrm{r}}^{(n)} = Q_n \mathscr{H}_{\mathrm{r}}$, where Q_n is a n-dimensional projection, then we can derive from $\mathfrak{T}(Q_n j) = \mathfrak{T}(j)$ the relations

$$\begin{aligned}(h, \delta/\delta j)\, \mathfrak{T}(j) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mathfrak{T}(j + \varepsilon h) - \mathfrak{T}(j)] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mathfrak{T}(Q_n j + \varepsilon Q_n h) - \mathfrak{T}(Q_n j)] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mathfrak{T}(j + \varepsilon Q_n h) - \mathfrak{T}(j)] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mathfrak{T}(Q_n j + \varepsilon h) - \mathfrak{T}(Q_n j)].\end{aligned}$$

Thus

$$\begin{aligned}(h, \delta/\delta j)\, \mathfrak{T}(j) &= (Q_n h, \delta/\delta j)\, \mathfrak{T}(Q_n j) \\ &= (Q_n h, \delta/\delta j)\, \mathfrak{T}(j) \\ &= (h, \delta/\delta j)\, \mathfrak{T}(Q_n j).\end{aligned} \tag{2.6}$$

Another important property of continuously differentiable tame functionals is the mean value theorem

$$\mathfrak{T}(j + h) = \mathfrak{T}(j) + (h, \delta/\delta j)\, \mathfrak{T}(j + \vartheta h), \quad \vartheta \in [0, 1], \tag{2.7}$$

which is in this case nothing but a basis-independent formulation of the mean value theorem for functions.

In Part I, Def. 3.1 we had introduced a domain $\mathscr{C}_{\mathfrak{D}}$ of one time continuously differentiable, tame functionals which should together with their first derivatives be majorized by certain positive functionals. These positive functionals, however, were not sufficiently characterized to justify all operations in the proof of Prop. 3.2. In this paper we make a special choice for the majorizing functionals which possesses all desired properties. Beside this it has proved advantageous to work with twice continuously differentiable functionals. Thus we introduce the following domain.

Definition 2.4. Let $\mathfrak{D}$ be dense in $\mathscr{H}_{\mathrm{r}}$. By $\mathscr{C}_{\mathfrak{D}}^{(2)}$ we denote the set of twice continuously differentiable functionals $\mathfrak{T}(j)$ which are based on subspaces $\mathscr{H}_{\mathrm{r}}^{(n)} = Q_n \mathscr{H}_{\mathrm{r}} \subset \mathfrak{D}$ and the second derivatives of which satisfy

$$|(h, \delta/\delta j)(g, \delta/\delta j)\, \mathfrak{T}(j)| \leqq K\, \|h\|_{\mathrm{r}}\, \|g\|_{\mathrm{r}} \exp[\alpha\, \|Q_n j\|_{\mathrm{r}}^2], \tag{2.8}$$

where $0 < \alpha < 1/4$.

Remark 1. The right-hand side of (2.8) is a square-integrable functional in contrast to the functional $K \exp[\alpha \|j\|_{\mathrm{r}}^2]$, $0 < \alpha < 1/4$, which is of infinite norm. By the mean value theorem we conclude from (2.8) observing (2.6)

$$(g, \delta/\delta j)\, \mathfrak{T}(j) = (g, \delta/\delta j)\, \mathfrak{T}(Q_n j) =: \mathfrak{T}_g(Q_n j) = \mathfrak{T}_g(0) + (Q_n j, \delta/\delta j)\, \mathfrak{T}_g(\vartheta Q_n j),$$

where $\vartheta \in [0, 1]$, of course, depends on j. Thus,

$$\begin{aligned}|\mathfrak{T}_g(Q_n j)| &\leqq |(g, \delta/\delta j)\, \mathfrak{T}(0)| + |(Q_n j, \delta/\delta j)(g, \delta/\delta j)\, \mathfrak{T}(\vartheta Q_n j)| \\ &\leqq K_1 \|g\|_{\mathrm{r}} + K_2 \|g\|_{\mathrm{r}}\, \|Q_n j\|_{\mathrm{r}} \exp[\alpha\, \|Q_n j\|_{\mathrm{r}}^2].\end{aligned}$$

Let us observe that to every $\beta > 0$ (how small it ever might be) there is a number $L > 0$ so that $|x| \leqq L \exp[\beta x^2]$. Choosing β as to satisfy $\alpha' = \alpha + \beta < 1/4$ and identifying $|x|$ with $\|Q_n j\|_r$ we obtain

$$|(g, \delta/\delta j)\, \mathfrak{T}(j)| \leqq K' \|g\|_r \exp[\alpha' \|Q_n j\|_r^2], \tag{2.9}$$

$0 < \alpha' < 1/4$, and in similar fashion

$$|\mathfrak{T}(j)| \leqq K'' \exp[\alpha'' \|Q_n j\|_r^2], \qquad 0 < \alpha'' < 1/4. \tag{2.10}$$

Remark 2. $\mathscr{C}_{\mathfrak{D}}^{(2)}$ contains not only the dense set $\mathscr{P}_{\mathscr{D}}$ consisting of all polynomials based on subspaces of $\mathscr{D}$ but also a lot of infinite series of such polynomials. That means that the particle-number operators N and $\bar{N}$ [cf. Part I, (2.16), (2.17)] are of infinite value on certain elements of $\mathscr{C}_{\mathfrak{D}}^{(2)}$. It is a remarkable fact that with the method of "the analysis in function space" the subsequent statements on observables can be established without difficulty for states of an infinite number of particles.

The distinguished role that play tame functionals in the representation theory of symmetries and observables is partly due to the fact that the operators in their functional arguments can be affected with finite dimensional projections. The strong consequences from this circumstance are shown in the following Lemma.

Lemma. Let B be a densely defined operator in $\mathscr{H}_r$ and Q_n a projection on a finite-dimensional subspace contained in the domain of definition of B. Then $Q_n B^+$ is a bounded operator which may be extended to the whole of $\mathscr{H}_r$ in the way that

$$Q_n B^+ j = Q_n B^+ Q'_m j \tag{2.11}$$

for all $j \in \mathscr{H}_r$, where Q'_m projects on $\mathscr{H}_r^{(m)} := BQ_n \mathscr{H}_r$, $m \leqq n$. If the sequence of densely defined operators B_t tends weakly to zero for $t \to 0$, then $Q_n B_t^+$ tends uniformly to zero, i.e., with respect to the operator-norm. Q_n is to project on a finite-dimensional subspace lying in the intersection $\bigcap_t \mathscr{D}[B_t]$.

Proof. Let us choose an orthonormal basis $\{e_\nu\}_1^\infty$, $e_\nu \in \mathscr{D}[B]$ so that $Q_n = \sum_{\nu=1}^{n} e_\nu(e_\nu, \cdot)$. We define the extension of $Q_n B^+$ explicitly through

$$Q_n B^+ j := \sum_{\nu=1}^{n} e_\nu (B e_\nu, j), \qquad j \in \mathscr{H}_r. \tag{2.12}$$

In the case that $j \in \mathscr{D}[B^+]$ (2.12) clearly coincides with the operator-product $Q_n B^+$. Now

$$\|Q_n B^+ j\|_r \leqq \left(\sum_{\nu=1}^{n} \|B e_\nu\|_r\right) \|j\|_r = M \|j\|_r$$

which proves the boundedness of the extension $Q_n B^+$. If Q'_m projects on $BQ_n\mathscr{H}_r$, then

$$Q_n B^+ Q'_m j = \sum_{\nu=1}^{n} e_\nu (B e_\nu, Q'_m j) = \sum_{\nu=1}^{n} e_\nu (Q'_m B e_\nu, j) = \sum_{\nu=1}^{n} e_\nu (B e_\nu, j) = Q_n B^+ j.$$

Let B_t tend weakly to zero for $t \to 0$. Then

$$\|Q_n B_t^+\|_r = \sup_{\|f\|_r = 1} \|Q_n B_t^+ f\|_r = \sup_{\|f\|_r=1} \left\|\sum_{\nu=1}^{n} e_\nu (B_t e_\nu, f)\right\|_r \leqq \sup_{\|f\|_r=1} \sum_{\nu=1}^{n} |(B_t e_\nu, f)| \to 0,$$

since the last expression tends to zero for all $f \in \mathscr{H}_r$. Q.E.D.

Before entering into the discussion of the infinitesimal generators we have to supplement an improved version of the Prop. 3.2 of Part I for the new domain $\mathscr{C}_{\mathfrak{D}}^{(2)}$, that is, we have to show the strong continuity of $\pi(U_t)$ on $\mathscr{C}_{\mathfrak{D}}^{(2)}$ provided that U_t is a weakly continuous operator-function on the dense domain $\mathscr{D} \in \mathscr{H}_r$. For simplicity we again restrict the investigation to the point $t = 0$, where U_t is assumed to be equal to unity. We directly estimate the norm difference

$$\begin{aligned} \mathrm{d}(t) &:= \|\pi(U_t)\, \mathfrak{T}(j) - \mathfrak{T}(j)\| = \|\mathfrak{T}(Q_n U_t^+ j) - \mathfrak{T}(Q_n j)\| \\ &= \|(Q_n(U_t^+ - 1) j, \delta/\delta j)\, \mathfrak{T}(Q_n j + \vartheta Q_n (U_t^+ - 1) j)\| \end{aligned}$$

where $\mathfrak{T} \in \mathscr{C}_{\mathfrak{D}}^{(2)}$ is assumed to be based on $Q_n \mathscr{H}_{\mathrm{r}}$ and the mean value theorem had been applied. Let $\bar{Q}^t_{n'}$ project on the n'-dimensional subspace $(U_t - 1) Q_n \mathscr{H}_{\mathrm{r}}$, $n' \leqq n$. By Q^t_m we denote the projection on the smallest subspace containing $Q_n \mathscr{H}_{\mathrm{r}} \cup \bar{Q}^t_{n'} \mathscr{H}_{\mathrm{r}}$, so that $m \leqq 2n$. Observing (2.11) and $Q_n Q^+_m = Q_n$, $\bar{Q}^t_{n'} Q^t_m = \bar{Q}^t_{n'}$, we see that in $\mathrm{d}(t)$ j may be replaced everywhere by $Q^t_m j$. Application of (2.9) gives

$$\mathrm{d}(t) \leqq K' \| \, \| Q_n (U^+_t - 1) Q^t_m j \|_{\mathrm{r}} \exp[\alpha \, \| Q_n (1 + \vartheta (U^+_t - 1)) Q^t_m j \|^2_{\mathrm{r}}] \, \| \,.$$

Since in virtue of the Lemma $\| Q_n (U^+_t - 1) \|_{\mathrm{r}}$ tends to zero for $t \to 0$ and since $\| Q_n \|_{\mathrm{r}} < 1$, there is a t_0 so that

$$\begin{aligned} \| Q_n (1 + \vartheta (U^+_t - 1)) Q^t_m j \|_{\mathrm{r}} &\leqq \| Q_n Q^t_m j \|_{\mathrm{r}} + \vartheta \, \| Q_n (U^+_t - 1) Q^t_m j \|_{\mathrm{r}} \\ &\leqq (\| Q_n \|_{\mathrm{r}} + \| Q_n (U^+_t - 1) \|_{\mathrm{r}}) \, \| Q^t_m j \|_{\mathrm{r}} \leqq \| Q^t_m j \|_{\mathrm{r}} \end{aligned}$$

for all $t < t_0$. Thus

$$\mathrm{d}(t) \leqq K' \, \| Q_n (U^+_t - 1) \|_{\mathrm{r}} \, \| \exp[\alpha \, \| Q^t_m j \|^2_{\mathrm{r}}] \, \| Q^t_m j \|_{\mathrm{r}} \| \,.$$

Choosing a β with $\alpha' = \alpha + \beta < 1/4$ there is a L so that $\| Q^t_m j \|_{\mathrm{r}} \leqq L \exp[\beta \, \| Q^t_m j \|^2_{\mathrm{r}}]$. Hence

$$\begin{aligned} \mathrm{d}(t) &\leqq K' L \, \| Q_n (U^+_t - 1) \|_{\mathrm{r}} \, \| \exp[\alpha' \, \| Q^t_m j \|^2_{\mathrm{r}}] \| = K' L \, \| Q_n (U^+_t - 1) \|_{\mathrm{r}} \, \{ \textstyle\int \exp[(2\alpha' - \frac{1}{2}) \sum_{\mu=1}^{m} x^2_\mu] \prod_{\mu=1}^{m} \mathrm{d}x_\mu / \sqrt{2\pi} \}^{1/2} \\ &= K' L \, \| Q_n (U^+_t - 1) \|_{\mathrm{r}} \, (1 - 4\alpha')^{-m/2} \leqq K' L \, \| Q_n (U^+_t - 1) \|_{\mathrm{r}} (1 - 4\alpha')^{-n} \end{aligned}$$

for all $t < t_0$. Thus $\mathrm{d}(t) \to 0$ for $t \to 0$.

We now apply similar techniques to the real subject of this paper.

Proposition 2.1. Let A be defined on the dense domain $\mathscr{D} \subset \mathscr{H}_{\mathrm{r}}$ and generate the weakly continuous semigroup U_t, i.e., $A = \lim_{t \to 0} (1/t) [U_t - 1]$ in the weak topology. Then the limit

$$\sigma(A) = \lim_{t \to 0} (1/t) [\pi (U_t) - 1] \tag{2.13}$$

exists in the strong topology on $\mathscr{C}_{\mathfrak{D}}^{(2)} \subset \mathscr{L}^2 (\mathscr{H}_{\mathrm{r}}, \gamma)$. Moreover, for each $\mathfrak{T} \in \mathscr{C}_{\mathfrak{D}}^{(2)}$ which is based on $Q_n \mathscr{H}_{\mathrm{r}}$ we have the explicit representation

$$\sigma(A) \, \mathfrak{T}(j) = (Q_n A^+ j, \delta / \delta j) \, \mathfrak{T}(j) \,. \tag{2.14}$$

Proof. Since $(1/t) [U_t - 1] - A$ tends weakly to zero for $t \to 0$ we know from the Lemma that $Q_n (1/t) [U^+_t - 1]$ tends to $Q_n A^+$ in the uniform operator topology. Defining

$$Q_n R^+_t = Q_n (U^+_t - 1) - t Q_n A^+ \qquad \text{we have} \qquad Q_n U^+_t = Q_n + t Q_n A^+ + Q_n R^+_t \,,$$

where $\lim_{t \to 0} \| Q_n R^+_t \|_{\mathrm{r}} / t = 0$. Let us consider the normdifference

$$\begin{aligned} \mathrm{d}(t) := &\; \| (1/t) [\mathfrak{T} (Q_n U^+_t j) - \mathfrak{T} (Q_n j)] - (Q_n A^+ j, \delta / \delta j) \, \mathfrak{T} (Q_n j) \| \\ = &\; \| ((Q_n A^+ + Q_n R^+_t / t) j, \delta / \delta j) \, \mathfrak{T} (Q_n j + \vartheta (t Q_n A^+ + Q_n R^+_t) j) \\ &\qquad - (Q_n A^+ j, \delta / \delta j) \, \mathfrak{T} (Q_n j) \| \qquad \text{[cf. (2.7)]} \\ \leqq &\; \| (Q_n A^+ j, \delta / \delta j) [\mathfrak{T} (Q_n j + \vartheta (Q_n A^+ t + Q_n R^+_t) j) - \mathfrak{T} (Q_n j) \| \\ &+ \| (Q_n (R^+_t / t) j, \delta / \delta j) \, \mathfrak{T} (Q_n j + \vartheta (t Q_n A^+ + Q_n R^+_t) j) \| \\ =: &\; \mathrm{d}_1 (t) + \mathrm{d}_2 (t) \,. \end{aligned}$$

Applying once more (2.7) one obtains

$$\mathrm{d}_1 (t) = \| (Q_n A^+ j, \delta / \delta j) (\vartheta (Q_n A^+ t + Q_n R^+_t) j, \delta / \delta j) \, \mathfrak{T} (Q_n j + \vartheta' \vartheta (t Q_n A^+ + Q_n R^+_t) j) \| \,, \quad \vartheta' \in [0, 1] \,.$$

By Q^t_m we denote the projection on the smallest subspace containing the union

$$Q_n \mathscr{H}_{\mathrm{r}} \cup A Q_n \mathscr{H}_{\mathrm{r}} \cup R_t Q_n \mathscr{H}_{\mathrm{r}} \,,$$

where $m \leqq 3n$. Taking into account (2.11) we may replace j by $Q^t_m j$ everywhere in $\mathrm{d}_1 (t)$. According to (2.8)

$$\begin{aligned} \mathrm{d}_1 (t) \leqq &\; K \, \| Q_n A^+ \|_{\mathrm{r}} (t \, \| Q_n A^+ \|_{\mathrm{r}} + \| Q_n R^+_t \|_{\mathrm{r}}) \\ &\cdot \| \exp[\alpha (\| Q_n \|_{\mathrm{r}} + \| Q_n A^+ \|_{\mathrm{r}} t + \| Q_n R^+_t \|_{\mathrm{r}})^2 \, \| Q^t_m j \|^2_{\mathrm{r}}] \, \| Q^t_m j \|^2_{\mathrm{r}} \| \,. \end{aligned}$$

Since $\| Q_n \|_r < 1$ and since $\| Q_n R_t^+ \|_r \to 0$ for $t \to 0$, we can determine a t_0 so that

$$\| Q_n \|_r + t \| Q_n A^+ \|_r + \| Q_n R_t^+ \|_r < 1$$

for all $t < t_0$. Choosing a $\beta > 0$, so that $\alpha' = \alpha + \beta < 1/4$, there exists a $L > 0$ satisfying

$$\| Q_m^t j \|_r^2 \leqq L \exp[\beta \| Q_m^t j \|_r^2] .$$

Thus, for $t < t_0$,

$$\begin{aligned} d_1(t) &\leqq K'(t \| Q_n A^+ \|_r + \| Q_n R_t^+ \|_r) \exp[\alpha' \| Q_m^t j \|_r^2] \\ &\leqq K'(t \| Q_n A^+ \|_r + \| Q_n R_t^+ \|_r)(1 - 4\alpha')^{-3n/2} \end{aligned}$$

showing that $d_1(t) \to 0$, for $t \to 0$.

Observing that in $d_2(t)$ occur the same operators as in $d_1(t)$ we again may replace everywhere j by $Q_m^t j$. Relation (2.9) leads then to

$$\begin{aligned} d_2(t) &\leqq K \| Q_n R_t^+ / t \|_r \| \exp[\alpha (\| Q_n \|_r + t \| Q_n A^+ \|_r + \| Q_n R_t^+ \|_r)^2 \| Q_m^t j \|_r^2] \| Q_m^t j \|_r^2 \| \\ &\leqq K' \| Q_n R_t^+ / t \|_r (1 - 4\alpha')^{-3n/2}, \qquad 0 < \alpha' < 1/4, \end{aligned}$$

so that $d_2(t) \to 0$, for $t \to 0$. Q.E.D.

Proposition 2.2. Let A be the generator of a weakly continuous semigroup in the same sense as in Prop. 2.1, and let A and A^+ be defined on the common dense domain $\mathscr{D} \subset \mathscr{H}_r$. Then we have for an arbitrary orthonormal basis $\{e_\mu\}_1^\infty$, $e_\mu \in \mathscr{D}$, on the domain $\mathscr{C}_{\mathfrak{D}}^{(2)}$ the representation

$$\sigma(A) = \sum_{\mu=1}^{\infty} (e_\mu, j)(A^+ e_\mu, \delta/\delta j) = \sum_{\mu=1}^{\infty} (A e_\mu, j)(e_\mu, \delta/\delta j), \tag{2.15, 16}$$

where the series converge in the strong topology in the functional Hilbert space.

Proof. Let $\mathfrak{T} \in \mathscr{C}_{\mathfrak{D}}^{(2)}$ be based on $Q_n' \mathscr{H}_r$, then in virtue of Prop. 2.1 we know that

$$\sigma(A)\, \mathfrak{T}(j) = (Q_n' A^+ j, \delta/\delta j)\, \mathfrak{T}(j) .$$

Denoting $Q_m = \sum_{\mu=1}^{m} e_\mu (e_\mu, \cdot)$ we consider the norm difference

$$\begin{aligned} d_m &= \| (Q_n' A^+ j, \delta/\delta j)\, \mathfrak{T}(Q_n' j) - (A^+ Q_m j, \delta/\delta j)\, \mathfrak{T}(Q_n' j) \| \\ &= \| (Q_n' A^+ j, \delta/\delta j)\, \mathfrak{T}(Q_n' j) - (Q_n' A^+ Q_m j, \delta/\delta j)\, \mathfrak{T}(Q_n' j) \| \\ &= \| ((Q_n' A^+ - Q_n' A^+ Q_m) j, \delta/\delta j)\, \mathfrak{T}(Q_n' j) \| . \end{aligned}$$

The projection on the smallest subspace containing $Q_n' \mathscr{H}_r \cup (1 - Q_m) A Q_n' \mathscr{H}_r$ is called Q_k^m, $k \leqq 2n$. Replacing j by $Q_k^m j$ we get from (2.9)

$$d_m \leqq K \| Q_n' A^+ \|_r \| 1 - Q_m \|_r \| \exp[\alpha \| Q_k^m j \|_r^2] \| Q_k^m j \|_r \| \leqq K' \| 1 - Q_m \|_r (1 - 4\alpha')^{-n}, \quad 0 < \alpha' < 1/4.$$

Thus $d_m \to 0$ for $m \to \infty$.

In the same fashion we treat the difference

$$\begin{aligned} d_m' &= \| (Q_n' A^+ j, \delta/\delta j)\, \mathfrak{T}(Q_n' j) - (Q_m A^+ j, \delta/\delta j)\, \mathfrak{T}_n'(Q_n' j) \| \\ &= \| ((Q_n' A^+ - Q_n' Q_m A^+) j, \delta/\delta j)\, \mathfrak{T}(Q_n' j) \| \\ &\leqq K \| Q_n' A^+ - Q_n' Q_m A^+ \|_r (1 - 4\alpha')^{-n}, \qquad 0 < \alpha' < 1/4 . \end{aligned}$$

Now

$$\| Q_n' A^+ - Q_n' Q_m A^+ \|_r = \sup_{\|f\|_r = 1} \| Q_n' A^+ f - Q_n' Q_m A^+ f \|_r = \sup_{\|f'\|_r = 1} \| Q_n' A f' - Q_n' Q_m A^+ f' \|_r ,$$

where $f' \in \mathscr{D}$, since $\mathscr{D}$ is dense in $\mathscr{H}_r$. Thus

$$\begin{aligned} \| Q_n' A^+ - Q_n' Q_m A^+ \|_r &= \sup_{\|f'\|_r = 1} \| \sum_{\nu=1}^{n} e_\nu' (A e_\nu', f') - \sum_{\nu=1}^{n} \sum_{\mu=1}^{m} e_\nu' (e_\nu', e_\mu)(A e_\mu, f') \|_r \\ &\leqq \sup_{\|f'\|_r = 1} \sum_{\nu=1}^{n} | (e_\nu' - Q_m e_\nu', A^+ f') | < \varepsilon, \qquad \text{for} \qquad m > m_0(\varepsilon). \end{aligned}$$

Hence $d'_m \to 0$, for $m \to \infty$. Thus we have shown

$$\sigma(A)\,\mathfrak{T}(j) = \lim_{m\to\infty} (A^+ Q_m\, j, \delta/\delta j)\,\mathfrak{T}(j) = \sum_{\mu=1}^{\infty} (e_\mu, j)(A^+ e_\mu, \delta/\delta j)\,\mathfrak{T}(j) \tag{2.15a}$$

and

$$\sigma(A)\,\mathfrak{T}(j) = \lim_{m\to\infty} (Q_m\, A^+ j, \delta/\delta j)\,\mathfrak{T}(j) = \sum_{\mu=1}^{\infty} (A\, e_\mu, j)(e_\mu, \delta/\delta j)\,\mathfrak{T}(j) \tag{2.16a}$$

where the limits are to be understood in the strong sence. Q.E.D.

Propositions 2.1 and 2.2 provide us with explicit representations of $\sigma(A)$ for the rather large class of densely defined operators A which generate weak semigroups. But it is easily seen that these representations are also meaningful for arbitrary, densely defined operators A. Therefore, we can consider (2.14) and (2.16) as an extension of the operatorfunction $\sigma(A)$ to the set of all densely defined operators. (2.15) is an extension of $\sigma(A)$ to the somewhat smaller set of those operators A which possess a densely defined adjoint A^+. In order to make the domain-discussions not too sophisticated we shall freely pick out from the various representations of $\sigma(A)$ that one which is most accommodated to the respective purposes.

Proposition 2.3. $\sigma(A)$ is homomorphic with respect to the commutator composition, i.e.,

$$\sigma([A, B]) = [\sigma(A), \sigma(B)] \tag{2.17}$$

on $C^{(2)}_{\mathfrak{D}}$, where A^+B^+ and B^+A^+ are assumed to be defined on the dense domain $\mathscr{D}$.

Proof. a) For $j \in \mathscr{D}$ we have the (point-wise) identity

$$([A, B]^+ j, \delta/\delta j)\,\mathfrak{T}(j) = [(A^+ j, \delta/\delta j), (B^+ j, \delta/\delta j)]\,\mathfrak{T}(j)\,. \tag{2.18}$$

For this we evaluate

$$\begin{aligned}(h, \delta/\delta j)(A\, j, \delta/\delta j)\,\mathfrak{T}(j) &= \lim_{\varepsilon_1,\varepsilon_2\to 0} (1/\varepsilon_1\varepsilon_2)\{\mathfrak{T}(j + \varepsilon_2 h + \varepsilon_1 A\, j + \varepsilon_2\varepsilon_1 h) - \mathfrak{T}(j + \varepsilon_2 h) - \mathfrak{T}(j)\}\\ &= \lim_{\varepsilon_1,\varepsilon_2\to 0} (1/\varepsilon_1\varepsilon_2)\{\mathfrak{T}(j + \varepsilon_2 h + \varepsilon_1 A\, j + \varepsilon_1\varepsilon_2 A\, h) - \mathfrak{T}(j + \varepsilon_2 h + \varepsilon_1 A\, j)\}\\ &\quad + \lim_{\varepsilon_1,\varepsilon_2\to 0} (1/\varepsilon_1\varepsilon_2)\{\mathfrak{T}(j + \varepsilon_2 h + \varepsilon_1 A\, j) - \mathfrak{T}(j + \varepsilon_2 h) - \mathfrak{T}(j)\}\,.\end{aligned}$$

According to the mean value theorem

$$\mathfrak{T}(j + \varepsilon_2 h + \varepsilon_1 A\, j + \varepsilon_1\varepsilon_2 A\, h) = \mathfrak{T}(j + \varepsilon_2 h + \varepsilon_1 A\, j) + [(\varepsilon_1\varepsilon_2 A\, h, \delta/\delta j)\,\mathfrak{T}](j + \vartheta\,\varepsilon_1\varepsilon_2 A\, h)\,,$$

so that

$$(h, \delta/\delta j)(A\, j, \delta/\delta j)\,\mathfrak{T}(j) = (A\, h, \delta/\delta j)\,\mathfrak{T}(j) + (h, \delta/\delta j)(g, \delta/\delta j)\,\mathfrak{T}(j)|_{g=A\,j}\,.$$

Thus

$$\begin{aligned}\sigma(A)\,\sigma(B)\,\mathfrak{T}(j) &= (A^+ j, \delta/\delta j)(B^+ j, \delta/\delta j)\,\mathfrak{T}(j)\\ &= (B^+ A^+ j, \delta/\delta j)\,\mathfrak{T}(j) + (h, \delta/\delta j)(g, \delta/\delta j)\,\mathfrak{T}(j)|_{h=A^+ j,\, g=B^+ j}\end{aligned}$$

and

$$\sigma(B)\,\sigma(A)\,\mathfrak{T}(j) = (A^+ B^+ j, \delta/\delta j)\,\mathfrak{T}(j) + (g, \delta/\delta j)(h, \delta/\delta j)\,\mathfrak{T}(j)|_{g=B^+ j,\, h=A^+ j}\,.$$

Taking into account the commutativity and linearity of the functional derivative we arrive at (2.18).

b) Let Q_n be an orthogonal sequence of projections and $Q_n\,\mathscr{H}_{\mathrm{r}} \subset \mathscr{D}$ for all n. In virtue of (2.15a) it holds

$$\sigma([A, B])\,\mathfrak{T}(j) = \lim_{m\to\infty} ([A, B]^+ Q_n\, j, \delta/\delta j)\,\mathfrak{T}(j)$$

[using a)]

$$= \lim_{m\to\infty} [(A^+ Q_n\, j, \delta/\delta j), (B^+ Q_n\, j, \delta/\delta j)]\,\mathfrak{T}(j) = [\sigma(A), \sigma(B)]\,\mathfrak{T}(j)\,,$$

since the limits in the commutator exist separately. Q.E.D.

The most interesting case is, of course, that A generates a continuous group of orthogonal transformations U_t. A then is anti-selfadjoint and $\mathscr{D}[A] = \mathscr{D}[A^+] = \mathscr{D}$ is dense in $\mathscr{H}_{\mathrm{r}}$, so that the representation theorems above are valid. $\pi(U_t)$ is a continuous group of unitary transformations in $\mathscr{L}^2(\mathscr{H}_{\mathrm{r}}, \gamma)$ and has also an anti-selfadjoint generator $\sigma(A)$ (see [7], p. 368). Consequently, for this special class of operators we have the *-homomorphism property

$$\sigma(A)^+ = \sigma(A^+). \tag{2.19}$$

It is instructive to verify the rather indirectly inferred Eq. (2.19) by means of the explicit series representations for $\sigma(A)$. Formulae (2.15) and (2.16) imply for a properly chosen sequence of projections Q_n

$$\sigma(A)\,\mathfrak{T}(j) = \lim_{m\to\infty}\ \lim_{n\to\infty} (Q_m A^+ Q_n j, \delta/\delta j)\,\mathfrak{T}(j)$$
$$= \lim_{n\to\infty}\ \lim_{m\to\infty} (Q_m A^+ Q_n j, \delta/\delta j)\,\mathfrak{T}(j) = \lim_{n\to\infty} (Q_n A^+ Q_n j, \delta/\delta j)\,\mathfrak{T}(j).$$

Thus, for $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{C}^{(2)}_{\mathfrak{D}}$

$$\langle\sigma(A)\,\mathfrak{T}_1, \mathfrak{T}_2\rangle = \langle\lim_{n\to\infty} (Q_n A^+ Q_n j, \delta/\delta j)\,\mathfrak{T}_1, \mathfrak{T}_2\rangle = \lim_{n\to\infty} \langle(Q_n A^+ Q_n j, \delta/\delta j)\,\mathfrak{T}_1, \mathfrak{T}_2\rangle$$
$$= \lim_{n\to\infty} \int\Big(\sum_{\mu,\nu=1}^{n} x_\nu A_{\nu\mu} \frac{\partial}{\partial x_\mu}\mathfrak{T}_1^*(x)\Big)\,\mathfrak{T}_2(x)\exp\Big[-\Big(\sum_{\nu=1}^{n} x_\nu^2\Big)\Big/2\Big]\prod_{\nu=1}^{n}\frac{dx_\nu}{\sqrt{2\pi}}$$

where $(e_\nu, A e_\mu) = A_{\nu,\mu}$, $(e_\nu, j) = x_\nu$. Since A is antiselfadjoint $A_{\nu\mu}$ is an antisymmetrical matrix and

$$\sum_{\nu,\mu=1}^{n} x_\nu A_{\nu\mu}\frac{\partial}{\partial x_\mu}\Big(\sum_{\nu=1}^{n} x_\nu^2\Big)\Big/2 = \sum_{\nu,\mu=1}^{n} x_\nu A_{\nu\mu} x_\mu = 0\,.$$

The differentiation of the exponential, therefore, gives zero and partial integration leads to

$$\langle\sigma(A)\,\mathfrak{T}_1, \mathfrak{T}_2\rangle = -\lim_{n\to\infty}\int\mathfrak{T}_1^*(x)\Big[\sum_{\nu,\mu=1}^{n} x_\nu A_{\nu\mu}\frac{\partial}{\partial x_\mu}\mathfrak{T}_2(x)\Big]d\gamma_n = -\langle\mathfrak{T}_1, \sigma(A)\,\mathfrak{T}_2\rangle,$$

which means $\sigma(A)^+ = -\sigma(A) = \sigma(-A) = \sigma(A^+)$.

If A is anti-selfadjoint we know that $-i\sigma(A)$ is a selfadjoint operator, but we don't know its exact domain of definition. For the representation of a physical observable it is as good to give a domain where $-i\sigma(A)$ is essentially-selfadjoint, cf. Chapter 4. We present such a domain which has special importance being the common, dense domain of several generators of a Lie-group.

Proposition 2.4. Let A be anti-selfadjoint in $\mathscr{H}_{\mathrm{r}}$. Then $-i\sigma(A)$ is essentially-selfadjoint on $\mathscr{C}^{(2)}_{\mathfrak{D}}$, where $\mathscr{D}$ is the domain of A.

Proof. Let A generate the group of orthogonal operators U_t, then $\mathscr{C}^{(2)}_{\mathfrak{D}}$ is stable against $\pi(U_t)$. For $\mathfrak{T} \in \mathscr{C}^{(2)}_{\mathscr{D}}$ we obtain from (2.8)

$$|[(h, \delta/\delta j)(g, \delta/\delta j)\,\pi(U_t)\,\mathfrak{T}](j)| = |(h, \delta/\delta j)(g, \delta/\delta j)\,\mathfrak{T}(U_t^+ j)|$$
$$= |[(U_t^+ h, \delta/\delta j)(U_t^+ g, \delta/\delta j)\,\mathfrak{T}](U_t^+ j)| \leqq K\,\|U_t^+ h\|_{\mathrm{r}}\,\|U_t^+ g\|_{\mathrm{r}}$$
$$\cdot\exp[\alpha\,\|Q_n U_t^+ j\|_{\mathrm{r}}^2] = K\,\|h\|_{\mathrm{r}}\,\|g\|_{\mathrm{r}}\exp[\alpha\,\|Q_n' j\|_{\mathrm{r}}^2]$$

where $Q_n' = U_t Q_n U_t^{-1}$. Since $\mathfrak{T}(U_t^+ Q_n' j) = \mathfrak{T}(U_t^+ j)$, Q_n' is the appropriate projection and (2.8) is valid for $\pi(U_t)\,\mathfrak{T}(j)$, too, and $\pi(U_t)\,\mathfrak{T}(j) \in \mathscr{C}^{(2)}_{U_t\mathscr{D}}$. According to a theorem in [5] (p. 308), $A U_t f = U_t A f$ and $A U_t^+ f = U_t^+ A f$, for $f \in \mathscr{D}$. Thus $U_t\mathscr{D} = \mathscr{D}$ and $\pi(U_t)\,\mathfrak{T}(j) \in \mathscr{C}^{(2)}_{\mathfrak{D}}$. Hence the assumptions of a theorem of Cooper[8] are satisfied, which tells us that the restriction of $-i\sigma(A)$ on $\mathscr{C}^{(2)}_{\mathfrak{D}}$ is essentially-selfadjoint.

Q.E.D.

We supplement our discussion by showing that in any case, even for bounded operators A, domain investigations are necessary for $\sigma(A)$ in the functional Hilbert space.

Proposition 2.5. If the domain of definition of $\sigma(A)$ is the whole of $\mathscr{L}^2(\mathscr{H}_{\mathrm{r}}, \gamma)$ then $A = 0$.

Proof. Let $A \neq 0$, then there is an $e_1 \in \mathscr{H}_{\mathrm{r}}$, $(e_1, e_1) = 1$, and $A e_1 \neq 0$. Consider the special square-integrable functional

$$\mathfrak{T}(j) = \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{\sqrt{n!}}\,He_n((e_1, j)),$$

which has the norm $\|\mathfrak{T}\|^2 = \sum_{n=1}^{\infty}(1/n^2) < \infty$. e_1 is completed to an orthonormal basis $\{e_\nu\}_1^\infty$ of $\mathscr{H}_\mathrm{r}$, and we denote $x_\nu = (e_\nu, j)$, $A_{\mu\nu} = (e_\mu, A e_\nu)$. Then

$$\sigma(A)\,\mathfrak{T}(j) = \sum_n \frac{1}{n}\frac{1}{\sqrt{n!}}\left(\sum_{\mu=1}^{\infty} x_\mu A_{\mu 1}\frac{\partial}{\partial x_1}\right) He_n(x_1).$$

Observing the recursion relations of the Hermitian polynomials (see [9], p. 105) we obtain

$$\left(\sum_{\mu=1}^{\infty} x_\mu A_{\mu 1}, \partial/\partial x_1\right) He_n(x_1) = n\left[\left(\sum_{\mu=2}^{\infty} x_\mu A_{\mu 1}\right) He_{n-1}(x_1) + A_{11}(He_n(x_1) + (n-1)\,He_{n-2}(x_1))\right] =: n\,G_n(x).$$

The orthogonality relations for the Hermitian polynomials lead to

$$\begin{aligned}\int G_n(x)\,G_m(x)\,\delta\gamma(x) &= \left(\sum_{\mu=2}^{\infty} A_{\mu 1}^2\right)\delta_{n,m}(n-1)! + A_{11}^2\, n!\,\delta_{n,m}\\ &\quad + A_{11}^2(m-1)!\,\delta_{n,m-2} + A_{11}^2(n-1)!\,\delta_{n-2,m} + A_{11}^2(m-1)^2(m-2)!\,\delta_{n,m}\\ &\geqq \left(\sum_{\mu=1}^{\infty} A_{\mu 1}^2\right)(n-1)!\,\delta_{n,m} = \|A e_1\|^2(n-1)!\,\delta_{n,m}.\end{aligned}$$

Thus,

$$\|\sigma(A)\,\mathfrak{T}(j)\|^2 = \sum_{n,m}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}\int G_n(x)\,G_m(x)\,\delta\gamma(x) \geqq \|A e_1\|^2 \sum_{n=1}^{\infty}\frac{1}{m} = \infty.$$

Therefore, the domain of definition of $\sigma(A)$ is not the whole of $\mathscr{L}^2(\mathscr{H}_\mathrm{r}, \gamma)$. Q.E.D.

Remark. Prop. 2.5 implies, that a one-parametric group of transformations $\pi(U_t)$ in the functional Hilbert space cannot be uniformly continuous since its generator is in any case unbounded (see [10], p. 614). Remarkable is also the fact that for the proof of Prop. 2.5 it is not necessary to assume $\mathscr{H}_\mathrm{r}$ to be of infinite dimension.

The preceding results can directly be applied to the infinitesimal generators of a Lie group as each such operator generates a one-parametric sub-group of that Lie group.

Proposition 2.6. Let A_l, $l = 1, 2, \ldots, s$, be a representation of a Lie algebra by anti-selfadjoint operators in $\mathscr{H}_\mathrm{r}$ which satisfy on a common, dense domain $\mathscr{D}$ the commutation relations

$$[A_l, A_m] = \sum_{n=1}^{s} C_{l,m}^n A_n.$$

Then the operators $\sigma(A_l)$ constitute a representation of the same algebra being anti-selfadjoint operators in $\mathscr{L}^2(\mathscr{H}_\mathrm{r}, \gamma)$. They are essentially anti-selfadjoint on the common, dense domain $\mathscr{C}_{\mathfrak{D}}^{(2)}$ [i.e. their restrictions $\sigma'(A_l)$ on $\mathscr{C}_{\mathfrak{D}}^{(2)}$ fulfill $\sigma'(A_l)^+ \subset -\sigma'(A)^{++}$] on which

$$[\sigma(A_l), \sigma(A_m)] = \sum_{n=1}^{s} C_{l,m}^n\,\sigma(A_n)$$

is valid.

Proof. All assertions can be easily derived from the foregoing propositions. Products such as $\sigma(A_l)\,\sigma(A_k)$ are well defined on $\mathscr{C}_{\mathfrak{D}}^{(2)}$ since the elements of $\mathscr{C}_{\mathfrak{D}}^{(2)}$ are twice differentiable. Q.E.D.

3. Representation of the Infinitesimal Generators of $\pi(U)$ and $\overline{\pi}_\lambda(U)$

In Part I we have introduced the functional integral-transformation

$$\boldsymbol{R}\,\mathfrak{T}(j) = \int \mathfrak{T}(j + i h)\exp[-(h, h)/2]\,\delta h/\sqrt{2\pi}$$

[cf. I, (2.12)], which transforms the polynomial basis of $\mathscr{L}^2(\mathscr{H}_r, \gamma)$ into an orthonormal basis system, and we have also defined the unitary transformation $\boldsymbol{V}\mathfrak{T}(j) = \exp[-(j,j)/2]\,\boldsymbol{D}_{\sqrt{2}}, \boldsymbol{D}_{\sqrt{2}}$ denoting a dilatation about the factor $\sqrt{2}$ [cf. I, (2.18) and (2.15)], where $\boldsymbol{V}$ connects the space $\mathscr{L}^2(\mathscr{H}_r, \gamma)$ with the space $\mathscr{L}^2(\mathscr{H}_r, \lambda)$. The transformed group-representing operator-functions were denoted by $\bar{\pi}(U) = \boldsymbol{R}\pi(U)\boldsymbol{R}^{-1}$ and $\pi_\lambda(U) = \boldsymbol{V}\boldsymbol{R}\pi(U)\boldsymbol{R}^{-1}\boldsymbol{V}^{-1}$. Since $\boldsymbol{R}$ had been investigated only on the dense domain $\mathscr{P}_\mathscr{D} \subset \mathscr{C}_\mathscr{D}$ we restrict the subsequent considerations to this smaller domain. It had been shown that $\bar{\pi}(U_t)$ and $\pi_\lambda(U_t)$ are strongly continuous on $\mathscr{P}_\mathscr{D}$ in the case that U_t is weakly continuous on $\mathscr{D}$. Thus we are able to make in analogy to Definition 2.1 the following definition.

Definition 3.1. Let A be the densely defined generator of a weakly continuous semigroup U_t in $\mathscr{H}_r$, then we denote by $\bar{\sigma}(A)$ resp. $\sigma_\lambda(A)$ those operator-functions which map A into the generator of $\bar{\pi}(U_t)$ resp. $\pi_\lambda(U_t)$.

From Definition 3.1 one can only conclude that $\bar{\sigma}(A)$ and $\sigma_\lambda(A)$ are densely defined, the specific domain of definition, however, is not known. Beside this, it is not trivial, e.g., that $\bar{\sigma}(A)$ is the $\boldsymbol{R}$-transformed of $\sigma(A)$, since $\boldsymbol{R}^{-1}$ (and perhaps also $\boldsymbol{R}$) is an unbounded operator in the functional Hilbert space. Yet it holds the following Proposition.

Proposition 3.1. Let A be defined on the dense domain $\mathscr{D} \subset \mathscr{H}_r$ and be the generator of the weak semigroup U_t. Then $\bar{\sigma}(A)$ resp. $\sigma_\lambda(A)$ are well defined on the domain $\mathscr{P}_\mathscr{D}$ resp. $\mathscr{P}^\lambda_\mathscr{D} := \boldsymbol{V}\mathscr{P}_\mathscr{D}$ and there the relations

$$\bar{\sigma}(A) = \boldsymbol{R}\,\sigma(A)\,\boldsymbol{R}^{-1} \tag{3.1}$$

and

$$\sigma_\lambda(A) = \boldsymbol{V}\boldsymbol{R}\,\sigma(A)\,\boldsymbol{R}^{-1}\boldsymbol{V}^{-1} \tag{3.2}$$

are valid.

Proof. We know $\boldsymbol{R}\mathscr{P}_\mathscr{D} = \mathscr{P}_\mathscr{D}$ (cf. proof of I, Prop. 2.3), i.e., for each $\mathfrak{T} \in \mathscr{P}_\mathscr{D}$ there is a $\mathfrak{T}' \in \mathscr{P}_\mathscr{D}$ so that $\mathfrak{T} = \boldsymbol{R}\mathfrak{T}'$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\pi}(U_t)\,\mathfrak{T}(j) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}\pi(U_t)\,\mathfrak{T}'(j) = \frac{\mathrm{d}}{\mathrm{d}t}\int \pi(U_t)\,\mathfrak{T}'(j+ih)\,\delta\gamma(h) = \frac{\mathrm{d}}{\mathrm{d}t}\langle 1, \pi(U_t)\,\mathfrak{T}'(j+ih)\rangle_h$$
$$= \langle 1, \sigma(A)\,\mathfrak{T}'(j+ih)\rangle_h = \boldsymbol{R}\,\sigma(A)\,\boldsymbol{R}^{-1}\,\mathfrak{T}(j),$$

where the continuity of the scalarproduct was used. (3.2) follows immediately from the boundedness of $\boldsymbol{V}$. Q.E.D.

Relations (3.1) and (3.2) imply that also $\bar{\sigma}(A)$ and $\sigma_\lambda(A)$ are additive operator-functions which leave the commutator composition invariant. Constructing a similar counterexample as in Prop. 2.5 one can show that $\bar{\sigma}(A)$ resp. $\sigma_\lambda(A)$ are defined on the whole of $\mathscr{L}^2(\mathscr{H}_r, \gamma)$ resp. of $\mathscr{L}^2(\mathscr{H}_r, \lambda)$ only if $A = 0$.

Since $\bar{\sigma}(1) = \bar{\boldsymbol{N}}$ and $[A, 1] = 0$ we have

$$[\bar{\boldsymbol{N}}, \bar{\sigma}(A)] = 0, \qquad \text{for all } A. \tag{3.3}$$

For anti-Hermitian A $\sigma(A)$, $\bar{\sigma}(A)$ and $\sigma_\lambda(A)$ coincide A being the generator of an orthogonal group U_t for which $\bar{\pi}(U_t) = \pi(U_t) = \pi_\lambda(U_t)$ is true. In contrast to the π-functions it is possible also in the general case to give closed expressions for $\bar{\sigma}(A)$ and $\sigma_\lambda(A)$. Those expressions are of interest because they form an alternative possibility for the representation of observables in the functional Hilbert space and because they are the generators of non-unitary transformations which may occur in physical considerations. Finally they have value in itself being intimately connected with various canonical operator pairs.

Proposition 3.2. The $\boldsymbol{R}$-transformation maps the canonical pair $(g, \delta/\delta j)$, (f, j) into the canonical pair

$$a(g) := \boldsymbol{R}(g, \delta/\delta j)\,\boldsymbol{R}^{-1} = (g, \delta/\delta j), \tag{3.4}$$
$$a^+(f) := \boldsymbol{R}(f, j)\,\boldsymbol{R}^{-1} = (f, j) - (f, \delta/\delta j)$$

where $g, h \in \mathscr{H}_c$, and it holds

$$[a(g)]^+ = a^+(g) \tag{3.5}$$

for $g \in \mathscr{H}_r$.

Proof. It is sufficient to show the above relations for monomials $\mathfrak{P}(k, j)$ which are associated with an orthonormal basis $\{e_\nu\}$ of $\mathscr{H}_\Gamma$, i.e., $\mathfrak{P}(k, j) = \prod_\nu (k_\nu!)^{-1/2}(e_\nu, j)^{k_\nu}$, $k = \{k_\nu\}$, having only a finite number of non-vanishing terms. The $\boldsymbol{R}$-transform of $\mathfrak{P}(k, j)$ had been denoted by $\mathfrak{H}(k, j)$ (cf. I, Prop. 2.3).

a)
$$\begin{aligned}\boldsymbol{R}(e_\mu, \delta/\delta j)\, \boldsymbol{R}^{-1}\, \mathfrak{H}(k, j) &= \boldsymbol{R}(e_\mu, \delta/\delta j)\, \mathfrak{P}(k, j)\\
&= \boldsymbol{R}\, k_\mu^{1/2}\, \mathfrak{P}(k'', j), \qquad k_\nu'' := k_\nu - \delta_{\mu\nu},\\
&= k_\mu^{1/2}\, \mathfrak{H}(k'', j)\\
&= \prod_{\nu \neq \mu} (k_\nu!)^{-1/2}\, He_{k_\nu}((e_\nu, j))\, k_\mu/\sqrt{k_\mu!}\, He_{k_\mu - 1}((e_\mu, j))\\
&= \prod_{\nu \neq \mu} (k_\nu!)^{-1/2}\, He_{k_\nu}((e_\nu, j))\, (k_\mu!)^{-1/2}\, (e_\mu, \delta/\delta j)\, He_{k_\mu}((e_\mu, j)) \qquad (\text{cf. } ^9, \text{ p. } 105)\\
&= (e_\mu, \delta/\delta j)\, \mathfrak{H}(k, j).\end{aligned}$$

For arbitrary $g = \sum_{\nu=1}^{\infty} x_\nu e_\nu$ and $\mathfrak{T} \in \mathscr{P}_{\{e_\nu\}} \subset \mathscr{C}^{(2)}_{\{e_\nu\}}$ ($\mathfrak{T}$ is a finite linear combination of the $\mathfrak{P}(k, j)$'s and also of the $\mathfrak{H}(k, j)$'s)

$$\Big\| \sum_{\nu=1}^{n} x_\nu (e_\nu, \delta/\delta j)\, \mathfrak{T} - (g, \delta/\delta j)\, \mathfrak{T} \Big\| \leqq K' \Big\| \sum_{\nu=1}^{n} x_\nu e_\nu - g \Big\|_\Gamma \to 0,$$

for $n \to \infty$ [cf. (2.9)]. Thus

$$\begin{aligned}\boldsymbol{R}(g, \delta/\delta j)\, \mathfrak{T}(j) &= \langle 1, \sum_{\nu=1}^{\infty} x_\nu (e_\nu, \delta/\delta j)\, \mathfrak{T}(j + ih) \rangle_h\\
&= \sum_{\nu=1}^{\infty} x_\nu \langle 1, (e_\nu, \delta/\delta j)\, \mathfrak{T}(j + ih) \rangle_h = (g, \delta/\delta j)\, \boldsymbol{R}\, \mathfrak{T}(j).\end{aligned}$$

b)
$$\begin{aligned}\boldsymbol{R}(f, j)\, \mathfrak{T}(j) &= \int (f, j + ih)\, \mathfrak{T}(j + ih)\, \delta\gamma(h)\\
&= (f, j)\, \boldsymbol{R}\, \mathfrak{T}(j) - i \int \mathfrak{T}(j + ih)\, (f, \delta/\delta j) \exp[-(h, h)/2]\, \delta(h/\sqrt{2\pi})\\
&= (f, j)\, \boldsymbol{R}\, \mathfrak{T}(j) + i \int (f, \delta/\delta h)\, \mathfrak{T}(j + ih)\, \delta\gamma(h)\\
&= [(f, j) - (f, \delta/\delta j)]\, \boldsymbol{R}\, \mathfrak{T}(j).\end{aligned}$$

c) Let $\mathfrak{T}$, $\mathfrak{S}$ be elements of $\mathscr{P}_{\mathscr{D}}$, $g \in \mathscr{H}_\Gamma$.

$$\begin{aligned}\langle (g, \delta/\delta j)\, \mathfrak{T}, \mathfrak{S} \rangle &= \int [(g, \delta/\delta j)\, \mathfrak{T}^*(j)]\, \mathfrak{S}(j) \exp[-(j, j)/2]\, \delta(j/\sqrt{2\pi})\\
&= \int \mathfrak{T}^*(j)\, [(-(g, \delta/\delta j) + (g, j))\, \mathfrak{S}(j)] \exp[-(j, j)/2]\, \delta(j/\sqrt{2\pi})\\
&= \langle \mathfrak{T}, [(g, j) - (g, \delta/\delta j)]\, \mathfrak{S}(j) \rangle. \qquad \text{Q.E.D.}\end{aligned}$$

By means of Prop. 3.2 one can justify rigorously a representation of the Hermitian polynomials[11], which mostly is written only in a formal manner. For,

$$\mathfrak{H}(k, j) = \boldsymbol{R} \prod_\nu (k_\nu!)^{-1/2} (e_\nu, j)^{k_\nu} = \prod_\nu (k_\nu!)^{-1/2} [\boldsymbol{R}(e_\nu, j)\, \boldsymbol{R}^{-1}]^{k_\nu}\, \boldsymbol{R} \cdot 1$$

$$\mathfrak{H}(k, j) = \prod_\nu (k_\nu!)^{-1/2}\, a^+(e_\nu)^{k_\nu} \cdot 1, \tag{3.6}$$

since $\boldsymbol{R} \cdot 1 = \int \delta\gamma(j) = 1$.

Proposition 3.3. Let A and A^+ be defined on the common dense domain $\mathscr{D} \subset \mathscr{H}_\Gamma$ and let $\{e_\nu\}$, $e_\nu \in \mathscr{D}$, be an orthonormal basis of $\mathscr{H}_\Gamma$. Then we have on $\mathscr{P}_{\mathscr{D}}$ the representation

$$\begin{aligned}\bar{\sigma}(A) &= \sum_{\nu=1}^{\infty} a^+(e_\nu)\, a(A^+ e_\nu) = \sum_{\nu=1}^{\infty} a^+(A e_\nu)\, a(e_\nu) = \sigma(A) - \sum_{\nu=1}^{\infty} (e_\nu, \delta/\delta j)(A^+ e_\nu, \delta/\delta j)\\
&=: (A^+ j, \delta/\delta j) - (A^+ \delta/\delta j, \delta/\delta j)\end{aligned} \tag{3.7}$$

where the infinite series converge in the strong sense and where the last expression is only a formal abbreviation. Moreover

$$\bar{\sigma}(A^+) = \bar{\sigma}(A)^+. \tag{3.8}$$

Proof. Let $\mathfrak{T}$, $\mathfrak{T}'$ be elements of $\mathscr{P}_{\mathscr{D}}$ and $\mathfrak{T} = \boldsymbol{R}\mathfrak{T}'$. According to (2.15) we have

$$\begin{aligned}\bar{\sigma}(A)\,\mathfrak{T}(j) &= \boldsymbol{R}\,\sigma(A)\,\mathfrak{T}'(j) = \boldsymbol{R}\sum_{\nu=1}^{\infty}(e_\nu, j)(A^+e_\nu, \delta/\delta j)\,\mathfrak{T}'(j)\\ &= \langle 1, \sum_\nu (e_\nu, j+ih)(A^+e_\nu, \delta/\delta j)\,\mathfrak{T}'(j+ih)\rangle_h\\ &= \sum_\nu \langle 1, (e_\nu, j+ih)(A^+e_\nu, \delta/\delta j)\,\mathfrak{T}'(j+ih)\rangle_h\\ &= \sum_\nu \boldsymbol{R}(e_\nu, j)\,\boldsymbol{R}^{-1}\boldsymbol{R}(A^+e_\nu, \delta/\delta j)\,\boldsymbol{R}^{-1}\mathfrak{T}(j)\\ &= \sum_\nu a^+(e_\nu)\,a(A^+e_\nu)\,\mathfrak{T}(j).\end{aligned}$$

Starting from (2.16) we obtain quite analogously

$$\bar{\sigma}(A)\,\mathfrak{T}(j) = \sum_\nu a^+(A\,e_\nu)\,a(e_\nu)\,\mathfrak{T}(j).$$

To obtain the third representation of (3.7) we have only to observe (3.4) and to note that the first series converges separately to $\sigma(A)$ proving the strong convergence of the second series. The fact that the orthonormal system is arbitrary if it lies in $\mathscr{D}$ justifies the basis-independent notation of (3.7).

In order to prove (3.8) we apply (3.5) to the first expression of (3.7) and thus realize that

(i) $$\langle\bar{\sigma}(A)^+\mathfrak{T}, \mathfrak{S}\rangle = \langle\mathfrak{T}, \sum_\nu a^+(e_\nu)\,a(A^+e_\nu)\,\mathfrak{S}\rangle = \sum_\nu\langle a^+(A^+e_\nu)\,a(e_\nu)\,\mathfrak{T}, \mathfrak{S}\rangle$$

for $\mathfrak{T}, \mathfrak{S} \in \mathscr{P}_{\mathscr{D}}$, and that

(ii) $$\langle\bar{\sigma}(A^+)\,\mathfrak{T}, \mathfrak{S}\rangle = \sum_\nu\langle a^+(A^+e_\nu)\,a(e_\nu)\,\mathfrak{T}, \mathfrak{S}\rangle$$

where the second form of (3.7) has been used. (We always employd the continuity of the scalar-product.)

Thus $\langle[\bar{\sigma}(A)^+ - \bar{\sigma}(A^+)]\,\mathfrak{T}, \mathfrak{S}\rangle = 0$ and since $\mathscr{P}_{\mathscr{D}}$ is dense it follows that $\sigma(A)^+\mathfrak{T} = \bar{\sigma}(A^+)\,\mathfrak{T}$, for $\mathfrak{T} \in \mathscr{P}_{\mathscr{D}}$. Q.E.D.

From the third equation in (3.7) it can be seen that $\bar{\sigma}(A) = \sigma(A)$ for anti-Hermitian A. Under certain assaumptions $\pi(U_t)$ can be written in the form $\exp[\sigma(A)t]$. This shows that (3.8) proofs the $*$-homomorphism property of $\bar{\pi}(U)$, too.

We now turn to the investigation of $\sigma_\lambda(A)$.

Proposition 3.4. The $\boldsymbol{V}$-transformation maps the cannonical pair $a(g)$, $a^+(f)$ into the cannonical pair

$$\begin{aligned}\mathscr{A}(g) &:= \boldsymbol{V}a(g)\,\boldsymbol{V}^{-1} = \frac{1}{\sqrt{2}}[(j, g) + (g, \delta/\delta j)]\\ \mathscr{A}^+(f) &:= \boldsymbol{V}a^+(f)\,\boldsymbol{V}^{-1} = \frac{1}{\sqrt{2}}[2(f, j) - (j, f) - (f, \delta/\delta j)]\end{aligned} \tag{3.9}$$

where $g, f \in \mathscr{H}_c$.

Proof. Let us calculate the commutators of $\boldsymbol{V} = \exp[-(j, j)/2]\,\boldsymbol{D}_{\sqrt{2}}$ with the multiplication and differentiation operators. First of all it is easily seen that

$$\begin{aligned}\boldsymbol{D}_\alpha(g, \delta/\delta j) &= (g, \delta/\delta j)(1/\alpha)\boldsymbol{D}_\alpha,\\ \boldsymbol{D}_\alpha(f, j) &= (f, j)\alpha\,\boldsymbol{D}_\alpha\end{aligned} \tag{3.10}$$

for arbitrary real α. Cautiousness is in order in commuting $\exp[-(j, j)/2]$ with the complex differentiation operator. For, $\mathfrak{T}(j)$ has to be extended to complex j to give $(g, \delta/\delta j)\,\mathfrak{T}(j)$ a meaning in the case that $g \in \mathscr{H}_c$, and this extension must lead to a holomorphic functional. Thus the holomorphic complex extensions of (j, f) and (j, j) are (j^*, f) and (j^*, j). For the subsequent calculations we make the convention that during the complex differentiation-process $\mathfrak{T}(j)$ is to be replaced by its holomorphic complex extension (if it exists). After the differentiation has been performed j is again chosen as a real vector. In this sense the following formulas are valid.

$$(g, \delta/\delta j)(j, j)/2 = \tfrac{1}{2}[(j^*, g) + (g^*, j)] = (j, g); \tag{3.11}$$

$$(g, \delta/\delta j)(j, f) = (g, \delta/\delta j)(j^*, f) = (g^*, f), \tag{3.12}$$

$$\exp[-(j, j)/2](g, \delta/\delta j) = [(j, g) + (g, \delta/\delta j)]\exp[-(j, j)/2]. \tag{3.13}$$

Finally

$$\boldsymbol{V}(g, \delta/\delta j)\,\boldsymbol{V}^{-1} = \frac{1}{\sqrt{2}}[(j, g) + (g, \delta/\delta j)]$$

and

$$\boldsymbol{V}[(f, j) - (f, \delta/\delta j)]\,\boldsymbol{V}^{-1} = \frac{1}{\sqrt{2}}[2(f, j) - (j, f) - (f, \delta/\delta j)].$$

Q.E.D.

It is easy to see that according the above rules for complex differentiation

$$\mathscr{A}(g)\exp[-(j,j)/2]=0\,, \tag{3.14}$$
$$\mathscr{A}^+(g)\exp[-(j,j)/2]=\sqrt{2}\,(g,j)\exp[-(j,j)/2]\,,$$

and

$$[\mathscr{A}(g),\mathscr{A}^+(f)]=(f,g)\,, \tag{3.15}$$

for arbitrary $g, f \in \mathscr{H}_c$. Since (3.14) and (3.15) ensue also directly from transforming the corresponding relations for $(g, \delta/\delta j)$ and (f, j) by the operator $\boldsymbol{VR}$ and from the observation that

$$\boldsymbol{V}\cdot 1=\exp[-(j,j)/2]$$

they confirm the consistency of our complex differentiation rules.

If we choose an orthonormal system $\{e_\nu\}$ of $\mathscr{H}_r$, $\mathscr{A}(e_\nu)$, $\mathscr{A}^+(e_\nu)$ go over to the quantities $\mathscr{A}_\nu$, $\mathscr{A}_\nu^+$ of [4]. We also obtain an explicit expression for the basis-elements $\mathfrak{J}(k,j)$ of $\mathscr{L}^2(\mathscr{H}_r,\lambda)$ [cf. Part I, (2.18)].

$$\mathfrak{J}(k,j)=\boldsymbol{V}\mathfrak{H}(k,j)=\prod_\nu (k_\nu!)^{-1/2}[\boldsymbol{V}a^+(e_\nu)\boldsymbol{V}^{-1}]^{k_\nu}\boldsymbol{V}\cdot 1$$
$$=\prod_\nu (k_\nu!)^{-1/2}\mathscr{A}^+(e_\nu)\exp[-(j,j)/2] \tag{3.16}$$

which makes visible the connection with[4], (I.5) (there is a deviation in the normalization factor).

Proposition 3.5. Let A and A^+ be defined on the common dense domain $\mathscr{D}\in\mathscr{H}_r$ and let $\{e_\nu\}$, $e_\nu\in\mathscr{D}$, be an orthonormal basis of $\mathscr{H}_r$. Then we have on $\mathscr{P}_{\mathfrak{D}}^\lambda := \boldsymbol{V}\mathscr{P}_{\mathscr{D}}$ the representation

$$\sigma_\lambda(A)=\sum_{\nu=1}^{\infty}\mathscr{A}^+(e_\nu)\,\mathscr{A}(A^+e_\nu)$$
$$=\sum_{\nu=1}^{\infty}\mathscr{A}^+(A\,e_\nu)\,\mathscr{A}(e_\nu)\,, \tag{3.17}$$

where the series converge in the strong topology.

If A is bounded and of finite trace

$$\sigma_\lambda(A)=\tfrac{1}{2}[(A^+j,\delta/\delta j)-(A\,j,\delta/\delta j)+$$
$$+(j,A\,j)-(A^+\delta/\delta j,\delta/\delta j)-T_r A]\,.$$

In any case we have

$$\sigma_\lambda(A^+)=\sigma_\lambda(A)^+\,. \tag{3.19}$$

Proof. (3.17) is simply the $\boldsymbol{V}$-transformed relation of (3.7). No convergence problems arise since $\boldsymbol{V}$ is bounded. Using (3.9) and the notation of Prop. 3.4 we arrive at (3.18) where the single terms make sense for its own. If A has no finite trace one must be cautious in splitting off the summands in (3.17). Thus $\sum_\nu (e_\nu,\delta/\delta j)(A^+e_\nu,\delta/\delta j)$ does not converge on $\mathscr{P}_{\mathfrak{D}}^\lambda$ in spite of the fact that it converges on $\mathscr{P}_{\mathfrak{D}}$, since the exponential gives rise to a trace-expression. In virtue of the unitarity of $\boldsymbol{V}$ (3.19) is an immediate consequence of (3.8). Q.E.D.

4. Applications to Quantum Field Theory

In the preceding chapters of this paper and of Part I we have investigated certain representations of symmetry groups and their infinitesimal generators in quite general terms. These representations, however, were induced by the functional formulation of quantum field theory, and we discuss now some consequences of our results in this specific context.

Our starting point for the symmetry representations in the functional Hilbert space had been throughout all chapters a group representation in the real Hilbert space $\mathscr{H}_r$. The choice of a real test function space corresponds to a field theory of an Hermitian field operator. Also in the case that the original theory was formulated by means of a non-Hermitian field it exists an equivalent Hermitian version of this theory[12,13]. In the latter the combinatorics of a perturbation expansion and certain integral-equations are simplified[12]. Beside this, there are also entirely mathematical arguments for the Hermitian version, since the functional integration theory is only fully developed for a real integration space. To perform the transition from the non-Hermitian to the Hermitian version we investigate the connection between complex and real group representations.

Let $\mathscr{H}'$ be a complex Hilbert space and j', h' elements of $\mathscr{H}'$ for which the scalarproduct is denoted by $(j',h')_c$. As each j' may be decomposed into its real and imaginary part $j'=j_1+ij_2$ the whole space $\mathscr{H}'$ can be split into $\mathscr{H}'=\mathscr{H}_\varrho+i\mathscr{H}_\varrho$, where $\mathscr{H}_\varrho$ is a real space. We now map $\mathscr{H}'$ into the direct sum $\mathscr{H}_\varrho\oplus\mathscr{H}_\varrho=:\mathscr{H}_r$, which is again a real space, defining the transformation

$$Z\,j'=\begin{pmatrix} j_1\\ j_2\end{pmatrix}=:j\,. \tag{4.1}$$

The scalar product is given by

$$(j,h)=(j_1,h_1)_c+(j_2,h_2)_c=\mathrm{Re}\,(j',h')_c\,. \tag{4.2}$$

Also the operators A' in $\mathscr{H}'$ can be decomposed into their real and imaginary components according to $A' = A_1 + i A_2$. The Z-transformed operator has the form

$$Z A' Z^{-1} = \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} =: A\,, \tag{4.3}$$

where the A_i, $i = 1, 2$, map $\mathscr{H}_\varrho$ into $\mathscr{H}_\varrho$. Thus

$$Z\, i\, Z^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: \eta\,, \tag{4.4}$$

where 1 is the unit operator in $\mathscr{H}_\varrho$. It holds

$$Z A'^{+} Z^{-1} = A^{\mathrm{T}}\,, \tag{4.5}$$

A^{T} being the transposed operator of A.

Since Z is homomorphic with respect to the operator multiplication, the inverse of A' is mapped into the inverse of A, which implies together with (4.5) that a unitary operator in $\mathscr{H}'$ is transformed into an orthogonal operator in $\mathscr{H}_{\mathrm{r}}$. The identity

$$\| Z j' \| = (j_1, j_1)_{\mathrm{c}} + (j_2, j_2)_{\mathrm{c}} = \| j' \|_{\mathrm{c}}$$

ensues the continuity of Z. Since i commutes with all linear operators A' in $\mathscr{H}'$ the real matrix η commutes with all Z-transformed operatorrs A in $\mathscr{H}_{\mathrm{r}}$.

The mapping Z is only a reformulation of the procedure employed in [13] to gain real group representations. There is another equivalent method which seems to have been used in [12] for the symmetry transformations. For this we introduce a mapping $\hat{Z}$ defined by

$$\hat{Z} j' = \hat{Z}(j_1 + i j_2) = \varepsilon_1 \otimes j_1 + \varepsilon_2 \otimes j_2 =: \hat{j}\,, \tag{4.6}$$

where $\varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\varepsilon_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so that the r.h.s. of (4.6) is an element of $\hat{\mathbb{C}} \otimes \mathscr{H}_\varrho$, $\hat{\mathbb{C}}$ denoting the realization of the complex numbers by real matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. The $\hat{Z}$-transformed of an operator $A' = A_1 + i A_2$ is

$$\hat{Z} A' \hat{Z}^{-1} = \varepsilon_1 \otimes A_1 + \varepsilon_2 \otimes A_2\,. \tag{4.7}$$

Defining the inner product of $\hat{\mathbb{C}}$ through

$$(\varepsilon_\alpha, \varepsilon_\beta) = \delta_{\alpha\beta}$$

we have in the product space $\hat{\mathbb{C}} \otimes \mathscr{H}_\varrho$

$$(\hat{j}, \hat{h}) = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} (\varepsilon_\alpha, \varepsilon_\beta)(j_\alpha, h_\beta)_{\mathrm{c}} = (j_1, h_1)_{\mathrm{c}} + (j_2, h_2)_{\mathrm{c}}$$

which proves the isomorphism of $\mathscr{H}_\varrho \oplus \mathscr{H}_\varrho$ and $\mathbb{C} \otimes \mathscr{H}_\varrho$. We shall only make use of the first mapping Z.

We now assume that there is given a non-Hermitian Boson field operator $\Phi(j')$, where j' is an element of a complex test function space $\mathscr{S}$. The transition to an Hermitian field is performed in [12,13] in the way that

$$\Psi(x) = \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix},$$
$$\Phi_1(x) = \operatorname{Re} \Phi(x), \quad \Phi_2(x) = \operatorname{Im} \Phi(x)\,. \tag{4.8}$$

The corresponding smeared Hermitian field would read

$$\Psi(j) = \int \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} \cdot (j_1(x), j_2(x))\, \mathrm{d}x = \Phi_1(j_1) + \Phi_2(j_2) \tag{4.9}$$

where

$$j = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \in \mathscr{H}_{\mathrm{r}}\,.$$

To give (4.9) a real meaning the space on which $\Psi(j)$ acts had to be specified.

To formulate symmetry conditions for the original field $\Phi(j')$ one has to assume that the symmetry transformations are represented in the test function space $\mathscr{S}$. This relates not only to space-time transformations but also to certain internal symmetries such as the isospin group. For definiteness we consider as the basic symmetry group solely the Poincaré group $\mathscr{P}_+$. In addition to the just mentioned assumption, which is standard in general quantum field theory, we require that there exists in $\mathscr{S}$ a (weighted) scalar product invariant against transformations of $\mathscr{P}_+$. The completion of $\mathscr{S}$ with respect to this scalar product is the complex Hilbert space $\mathscr{H}'$ in which we have now a unitary, continuous representation $U'(\Lambda, a)$, $(\Lambda, a) \in \mathscr{P}_+$, of the Poincaré group. In $\mathscr{H}'$ should be specified a dense domain $\mathscr{D}'$ where the Hermitian generators A'_l, A'_k standing for either a P_i' or a M'_{ik} satisfy

$$[i A'_l, i A'_k] = \sum_{n=1}^{10} C^n_{lk}\, i A'_n\,.$$

C^n_{lk} are the well-known real structure constants of the Lie algebra belonging to $\mathscr{P}_+$. Performing the Z transformation we obtain an orthogonal, continuous representation $U(\Lambda, a)$ in the space $\mathscr{H}_{\mathrm{r}}$ and the anti-Hermitian infinitesimal generators ηA_l fulfilling

$$[\eta A_l, \eta A_k] = \sum_{n=1}^{10} C^n_{lk}\, \eta A_n$$

on the dense domain $\mathscr{D} = Z \mathscr{D}' \subset \mathscr{H}_{\mathrm{r}}$.

According to Chapter 1 of Part I the transition into the functional space is effected via the time-ordered expectation values of $\Psi(j)$. This leads uniquely to a representation $\boldsymbol{U}(\Lambda, a) = \pi(U(\Lambda, a))$ of $\mathscr{P}_+$ in the space of generating functionals. The functional scalarproduct which makes $\boldsymbol{U}(\Lambda, a)$ to a unitary representation is that constructed by means of the Gaussian measures. In the corresponding functional Hilbert space $\mathscr{L}^2(\mathscr{H}_\mathrm{r}, \gamma)$ the unitary operators $\boldsymbol{U}(\Lambda, a)$ constitute a continuous, highly reducibel representation of $\mathscr{P}_+$, and $\mathscr{L}^2(\mathscr{H}_\mathrm{r}, \gamma)$ can be decomposed into the direct sum

$$\sum_{n=0}^{\infty} \oplus \overline{\mathscr{L}}_n^2(\mathscr{H}_\mathrm{r}, \gamma)$$

of relative orthogonal subspaces invariant against $\boldsymbol{U}(\Lambda, a)$. As $U(\Lambda, a)$ are orthogonal in $\mathscr{H}_\mathrm{r}$ we know that in the basis system of the Hermitian polynomials $\boldsymbol{U}(\Lambda, a) = \boldsymbol{R} \boldsymbol{U}(\Lambda, a) \boldsymbol{R}^{-1}$ retains the same form. If we included also non-orthogonal transformations such as, e.g., dilatations into our considerations it seemed to be most natural to represent them by the aid of the operator function $\overline{\pi}(U)$ which is a *-homomorphism.

The infinitesimal generators of $\mathscr{P}_+$ are represented on the dense domain $\mathscr{C}_{\mathfrak{D}}^{(2)} \subset \mathscr{L}^2(\mathscr{H}_\mathrm{r}, \gamma)$ by means of the operators $\boldsymbol{A}_l = -i\sigma(\eta A_l)$ so that a one-parametric subgroup of $\mathscr{P}_+$ can be written as $\exp[i\boldsymbol{A}_l t_l]$. More explicitly we have on $\mathscr{C}_{\mathfrak{D}}^{(2)}$

$$\boldsymbol{P}_i = i \sum_{\mu=1}^{\infty} (e_\mu, j)(\eta\, P_i\, e_\mu, \delta/\delta j) = i \sum_{\mu=1}^{\infty} a^+(e_\mu)\, a(\eta\, P_i\, e_\mu)$$

and

$$\boldsymbol{M}_{ik} = i \sum_{\mu=1}^{\infty} (e_\mu, j)(\eta\, M_{ik}\, e_\mu, \delta/\delta j) = i \sum_{\mu=1}^{\infty} a^+(e_\mu)\, a(\eta\, M_{ik}\, e_\mu)$$

where $\{e_\mu\}_1^\infty$, $e_\mu \in \mathscr{D}$ is an orthonormal basis of $\mathscr{H}_\mathrm{r}$.

On $\mathscr{C}_{\mathfrak{D}}^{(2)}$ the infinitesimal generators $\boldsymbol{A}_l$ are not only Hermitian but even essentially selfadjoint. This is important to know because only a self-adjoint operator corresponds to a physical observable, since only in this case the spectral theorem is valid, so that the set of (generalized) eigenvectors is complete. An essentially selfadjoint operator can be uniquely extended to a selfadjoint one and thus determines the physical observable also completely. If we were only able to give a domain on which the $\boldsymbol{A}_l$ are Hermitian we could not say if they correspond to one or to several or to none physical observable in the precise quantum mechanical sense.

The observables being fixed by explicit construction we can proceed to characterize the states in the functional Hilbert space without relating any more to the field-theoretic Hilbert space. For a complete identification of a state one needs not only the elements of the Lie algebra but also some elements of the so-called enveloping algebra, where the algebra product is in a representation realized by the operator product. In our relativistic functional theory we must consider beside the $\boldsymbol{P}_i$ and $\boldsymbol{M}_{ik}$ the operators $\boldsymbol{P}^2 = \boldsymbol{P}_l \boldsymbol{P}^l$ and $\boldsymbol{W} = \boldsymbol{\Gamma}_l \boldsymbol{\Gamma}^l$, where $\boldsymbol{\Gamma}_l = \frac{1}{2} \varepsilon_{lkrs} \boldsymbol{P}^k \boldsymbol{M}^{rs}$. Since the operator function $\sigma(A)$ is homomorphic only with respect to the Lie algebra and not with respect to the enveloping algebra $\boldsymbol{P}^2$ and $\boldsymbol{W}$ are not the σ-transformed of the respective quantities in $\mathscr{H}_\mathrm{r}$. As $\boldsymbol{P}^2$ and $\boldsymbol{W}$ commute with all elements of the Lie algebra they are multiples of the identity in every irreducible representation of $\mathscr{P}_+$. The irreducibel parts of $\boldsymbol{U}(\Lambda, a)$ are thus restricted to subspaces of the $\overline{\mathscr{L}}_n^2(\mathscr{H}_\mathrm{r}, \gamma)$ in which $\boldsymbol{P}^2$ and $\boldsymbol{W}$ assume constant values, determining the mass and the spin of a particle. To characterize particular states a complete set of commuting observables is required, i.e., such a set, that all other observables commuting with this set are functions of the former. Using only the abstract commutation rules of the Lie algebra one knows that $\boldsymbol{P}_l$ and $\boldsymbol{\Gamma}_3$ form a maximal set of commuting operators which has the additional property of being translation invariant. This is, however, not a complete set of observables as the Hermitian operators $\overline{\boldsymbol{N}} = \overline{\sigma}(1)$ and

$$i\boldsymbol{N}' := i\,\overline{\sigma}(\eta) = i\,\sigma(\eta)$$

which are different from the unit-operator in $\mathscr{L}^2(\mathscr{H}_\mathrm{r}, \gamma)$ commute with all the $\boldsymbol{P}_i$ and $\boldsymbol{\Gamma}_3$ and are not expressible by functions of the latter. (Remember that η commutes with all the A_l and that $\overline{\sigma}(\cdot)$ leaves the commutator invariant.) $\overline{\boldsymbol{N}}$ may be interpreted as the particle-number operator.* $i\boldsymbol{N}'$ replaces the charge-operator of a complex quantum theory. In a theory which deals with the strong interaction only $i\boldsymbol{N}'$ may be identified with the baryon-number[12]. In both interpretations $\overline{\boldsymbol{N}}$ belongs to the set of observables

* This interpretation is only reasonable in the case that $\overline{N}$ commutes with the interaction.

while $i\,\boldsymbol{N}'$ generates a superselection rule. No state occuring in nature is a superposition of eigen-vectors to $i\,\boldsymbol{N}'$ which possess different eigen-values. If $\overline{\boldsymbol{N}}$, $\boldsymbol{P}_i$ and $\boldsymbol{\Gamma}_3$ constitute really a complete set of observables depends on the specific theory. In the Hermitian spinor theory of [12] an isospin-observable has to be included.

Let us remark that another, very natural representation of the infinitesimal generators namely $\bar{\sigma}(A_l)$ is excluded by our argumentation. The $\bar{\sigma}(A_l)$ are not the generators of $\boldsymbol{U}(\Lambda, a)$ which is uniquely induced by the transformation properties of the field $\Psi(j)$. On the other hand one could imagine still other mappings of the field-theoretic Hilbert space into the functional space such as, e.g.,

$$| B\rangle \mapsto \mathfrak{G}_{\mathrm{B}}(j) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle 0 |\, T\, \Psi(x_1) \cdots \Psi(x_n) \,|\, B\rangle\, a^+(x_1) \cdots a^+(x_n)\, \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n$$

where

$$a^+(x_i) = a^+(\delta(x_i - x)) = j(x_i) - \delta/\delta j(x_i)\,.$$

Many of these mappings, as the just mentioned, don't alter the structure of the symmetry representations.

Since we imbed the generating functionals into an Hilbert space each such map imposes a severe restriction on the time-ordered expectation values, requiring that they become small in some sense for increasing n. Thus every map and every functional Hilbert space is beside other things a concretization of the so-called third boundary condition of Heisenberg.

The special Hilbert space depends on the interaction of the underlying field theory. The preceding consideration show, however, that there is a quite general scheme of constructing the physical observables in this space. A state of this space can then be called physical, if it lies in the common domain of a complete set of observables. Prop. 2.5 tells us that the functional Hilbert space is in any case larger than the set of all physical states.

In practical calculations one never deals with normalizable states but with generalized eigenstates to the momentum operator $\boldsymbol{P}_\mu$. Especially in scattering calculations this is quite forcing since there seems to be no other simple method to incorporate the momentum conservation into the computations. On the other hand the requirement of unitary representations of the space-time translations ensues the use of translation invariant scalar products in the test function space, so that the functional norms of the momentum eigenstates diverge. The presented formalism seems to fail in this case. However, STUMPF has proposed in a recent work[14] a method how to eliminate the last mentioned divergencies in a systematic fashion maintaining at the same time all symmetry properties of the functional scalar product.

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